§6.1 Inner Products

Let u, v be vectors in $\mathbb{R}^n$. The <u>dot product</u> of u and v is defined as

$$u \cdot v = u^T v$$ (where T denotes transpose)

and the product on the right is matrix multiplication (makes sense since u^T is $1 \times n$ and v is $n \times 1$).

Notice the result is a number in $\mathbb{R}$.

In other words, if

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{then}$$

$$u \cdot v = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \sum_{i=1}^{n} u_i v_i$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

<u>Remark</u>

The dot product is an example of an <u>inner product</u>, a more general version of this. We'll develop this in §6.7.

Theorem

Using the summation from before, we may deduce the following:

Let u, v, w be vectors in $\mathbb{R}^n$ and c a scalar

a) $u \cdot v = v \cdot u$

b) $(u+v) \cdot w = u \cdot w + v \cdot w$

c) $(cu) \cdot v = c(u \cdot v)$

d) $u \cdot u \geq 0$ and $u \cdot u = 0$ if and only if $u = 0$.

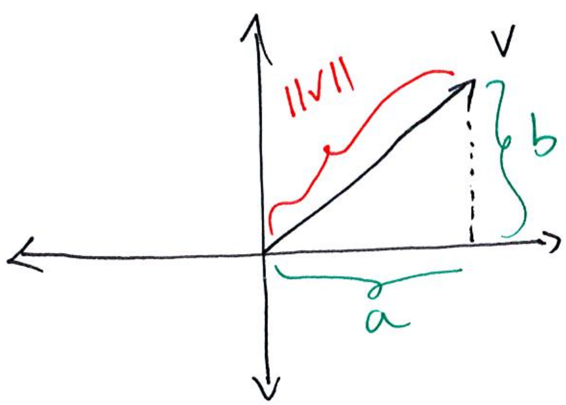
Definition

Let $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ be a vector in $\mathbb{R}^n$. The <u>length</u> or <u>norm</u> of v is $\|v\|$ where

$$\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Notice that by the Pythagorean theorem, this actually is the length of the vector in $\mathbb{R}^n$.

For example if $v = \begin{bmatrix} a \\ b \end{bmatrix}$ in $\mathbb{R}^2$, then



$$\|v\| = \sqrt{a^2 + b^2}$$

Properties

Let u, v be vectors in $\mathbb{R}^n$ and c a scalar

- $\|cv\| = |c| \cdot \|v\|$ absolute value
- $\|u+v\| \leq \|u\| + \|v\|$

Defn

A vector whose length is 1 is called a unit vector. For any vector (nonzero) v we can obtain a unit vector by scaling by $\frac{1}{\|v\|}$. If we write $u = \frac{v}{\|v\|}$, the u is called the normalization of v and is a unit vector in the same direction as v.

Example

Normalize $v = \begin{bmatrix} 4 \\ -1 \\ \sqrt{10} \\ 3 \end{bmatrix}$.

$$\|v\| = \sqrt{16+1+10+9} = \sqrt{36} = 6$$ (so v not a unit vector)

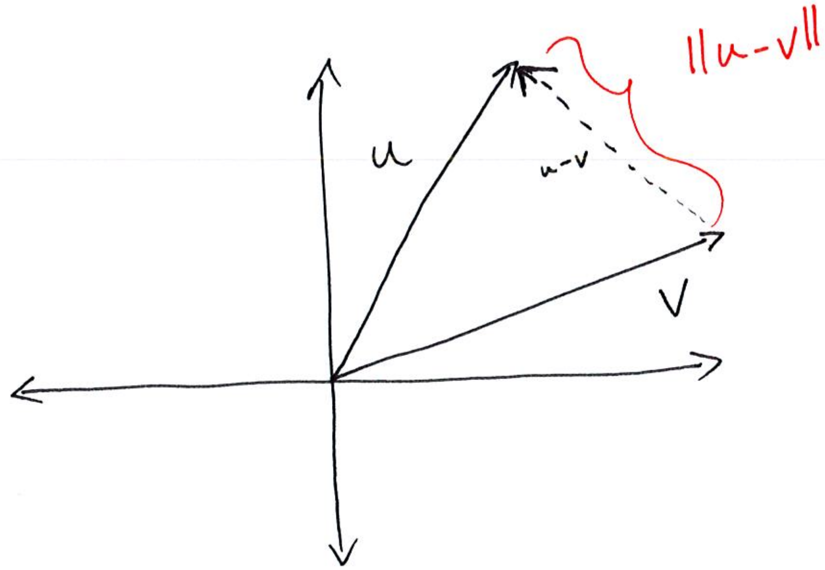
$$u = \frac{v}{\|v\|} = \begin{bmatrix} 2/3 \\ -1/6 \\ \sqrt{10}/6 \\ 1/2 \end{bmatrix}$$ is a unit vector.

(Verify $\|u\| = 1$ on your own!)

Definition

If u and v are vectors in $\mathbb{R}^n$, the distance between u and v is $\text{dist}(u,v) = \|u-v\|$.

measures distance between endpoints:



Orthogonality

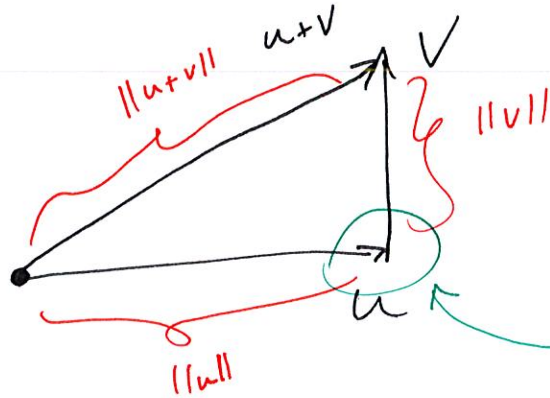
We say vectors u and v are orthogonal if $u \cdot v = 0$. In $\mathbb{R}^2$ this is the same as perpendicular so this is a higher-dimension generalization.

This generalization of right angles yields a more general version of a well-known theorem.

Pythagorean Theorem

Two vectors u and v are orthogonal if and only if

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$



all depends on this angle here